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Matrix product ground states for exclusion processes with parallel dynamics

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Abstract. We show in the example of a one-dimensional asymmetric exclusion process that stationary states of models with parallel dynamics may be written in a matrix product form. The corresponding algebra is quadratic and involves three different matrices. Using this formalism we prove previous conjectures for the equal-time correlation functions of the model.

During the last few years the study of one-dimensional reaction–diffusion models has become an object of increasing interest. These models describe stochastic processes far away from thermal equilibrium, for which reason their stationary probability distribution cannot generally be derived from an energy function. For this reason different techniques are needed in order to determine the stationary properties. An exact method which proved to be very successful is the so-called matrix product formalism [1–8]. This formalism can be regarded as a generalization of stationary states with a product measure in which products of numbers are replaced by products of non-commutative algebraic objects. By representing these objects in terms of matrices, the stationary state and all equal-time correlation functions can be derived exactly. So far this technique has been applied mainly to systems with sequential dynamics (continuous time evolution) where the stationarity of the state is related to an additive cancellation mechanism from site to site. However, many systems, for example traffic models [9], are defined by parallel dynamics (discrete time evolution) rather than sequential updates. Therefore it is of interest to find applications of the matrix product technique to systems with parallel dynamics. The present work discusses this problem in the example of a one-dimensional asymmetric exclusion process with parallel updates [10]. A modified matrix product formalism is presented in which a new multiplicative cancellation mechanism plays an essential role. The corresponding matrix algebra is derived and finite-dimensional representations are given. This makes it possible to prove exact results for the equal-time correlation functions which were given as conjectures in [10].

Let us first recall the matrix product formalism for one-dimensional reaction–diffusion models with sequential dynamics [3–8]. A two-state model with L sites and open boundary conditions is said to have a matrix product ground state if the stationary probability distribution $P_0(\tau_1, \tau_2, \dots, \tau_L)$ can be written as

$$P_0(\tau_1, \tau_2, \dots, \tau_L) = \frac{1}{Z} \langle W| \prod_{j=1}^{L} (\tau_j D + (1-\tau_j)E)\,|V\rangle \tag{1}$$

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where $\tau_j \in \{0, 1\}$ is the occupation number at site j. E and D are square matrices acting in an auxiliary space which may be either finite or infinite dimensional. The probabilities are the 'expectation values' $\langle W| \ldots |V\rangle$ of the matrix products normalized by the constant $Z = \langle W|(D+E)^L|V\rangle$. In the following it will be convenient to use a more formal notation in which equation (1) is written as a tensor product:

$$|P_0\rangle = \frac{1}{Z}\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle = Z^{-1}\langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes L} |V\rangle\,. \tag{2}$$

Here the tensor products act on pairs of matrices at neighbouring sites and the vector $|P_0\rangle$ represents the stationary probability distribution in configuration space, e.g. for $L = 2$ we have

$$\begin{aligned} |P_0\rangle &= \{P_0(0,0),\ \ P_0(0,1),\ \ P_0(1,0),\ \ P_0(1,1)\} \\ &= Z^{-1}\{\langle W|E^2|V\rangle,\ \langle W|ED|V\rangle,\ \langle W|DE|V\rangle,\ \langle W|D^2|V\rangle\}\,. \end{aligned} \tag{3}$$

When E and D are numbers (i.e. commutative objects) $|P_0\rangle$ is just an ordinary product measure state and all correlations are trivial. The matrix product technique can be seen as a generalization of product measure states by taking non-commutative objects E and D which leads to states with non-trivial correlations. If a matrix representation for these objects is known, various physical quantities like the local particle density

$$\langle \tau_j \rangle_L = \frac{\langle W|\, C^{j-1} D C^{L-j}\, |V\rangle}{\langle W|\, C^L\, |V\rangle} \qquad (C = D + E) \tag{4}$$

can be computed directly. Correlation functions are given by similar expressions in which C plays the role of a transfer matrix. However, a special mechanism is needed in order to ensure that the matrix product state in (2) is indeed a stationary one. For models with sequential dynamics this mechanism amounts to an additive cancellation from site to site. The time evolution of such models is described by a master equation $\frac{\mathrm{d}}{\mathrm{d}t}|P\rangle = -H|P\rangle$ with a time evolution operator $H = \sum_{j=1}^{L-1} h_{j,j+1} + h_1^{(L)} + h_L^{(R)}$, where $h_{j,j+1}$ is a 4×4 interaction matrix and $h^{(L)}$ and $h^{(R)}$ are 2×2 matrices for particle input and output at the ends of the chain. The usual ansatz for a cancellation mechanism is to assume that the application of the interaction matrix $h_{j,j+1}$ to a pair of matrices located at neighbouring sites results in a local divergence-like term on the right-hand side

$$h\left[\begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix}\right] = \begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} - \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} \tag{5}$$

where $\hat{E}$ and $\hat{D}$ are again matrices in the auxiliary space. In the sum $\sum_{j=1}^{L-1} h_{j,j+1}$ all these contributions cancel in the bulk of the chain. The remaining terms at the boundaries have to be cancelled by the surface fields for particle input and output:

$$\langle W|\, h^{(L)} \begin{pmatrix} E \\ D \end{pmatrix} = -\langle W| \begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} \qquad h^{(R)} \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle = \begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} |V\rangle\,. \tag{6}$$

Therefore if E, D, $\langle W|$ and $|V\rangle$ satisfy (5) and (6), the cancellation mechanism ensures that $H|P_0\rangle = 0$. It should be emphasized that this ansatz does not solve the problem. It only shifts the problem to the operator algebra defined by (5) and (6) and it depends on the properties of the physical system whether non-trivial solutions exist or not. However, there is a variety of systems with sequential dynamics where solutions are known. The simplest case $\hat{E} = \hat{D} = 0$ and its generalization to spin-one chains was considered in [3]. Another system which has been investigated in detail is the (asymmetric) exclusion process where $\hat{E} = -\hat{D} = 1$ [4]. Both cases lead to a quadratic algebra of two objects E and D [5].

Using similar methods matrix product ground states were found for particular three-state models [6]. Also, excited states can be described with a matrix ansatz [7] where $\hat{E} = -\hat{D}$ has to be chosen as a time-dependent matrix leading to a quadratic algebra of three different objects. By taking $\hat{E}$ and $\hat{D}$ as independent matrices (i.e. four independent objects), it was also possible to find the stationary state of particular models with particle reactions [8].

So far, interest has been focused mainly on stochastic models with continuous time evolution. However, similar techniques can be used for systems with parallel dynamics. A first example of this type was given in [1] where the transfer matrix for a deterministic model of directed animals on a strip was investigated. It is the aim of the present work to point out that there could be a broad spectrum of applications to reaction–diffusion models with parallel dynamics. For this purpose we consider a one-dimensional asymmetric exclusion process with parallel updates which was originally introduced by Schütz in [10]. In this model particles move on a one-dimensional lattice with $L = 2N$ sites and open boundaries. The bulk dynamics is deterministic and consists of two half time steps. In the first half time step, particles at odd positions move one step to the right provided that the neighbouring site to the right is empty. In the second half time step, the particles at even positions then move to the right in the same way. In addition particles are injected (removed) stochastically with rate α (β) at the left (right) boundary:

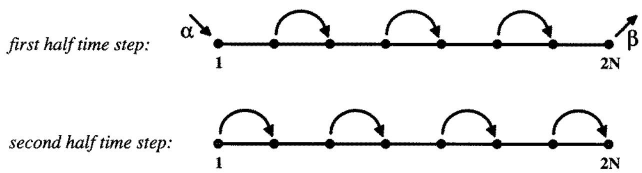

Figure 1.

The corresponding transfer matrix therefore consists of two factors $T = T_2 T_1$

$$\begin{aligned} T_1 &= \mathcal{L} \otimes \mathcal{T} \otimes \cdots \otimes \mathcal{T} \otimes \mathcal{R} = \mathcal{L} \otimes \mathcal{T}^{\otimes(N-1)} \otimes \mathcal{R} \\ T_2 &= \mathcal{T} \otimes \mathcal{T} \otimes \cdots \otimes \mathcal{T} = \mathcal{T}^{\otimes N} \end{aligned} \tag{7}$$

where $\mathcal{T}$, $\mathcal{L}$ and $\mathcal{R}$ are the matrices for hopping, particle input and output:

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L} = \begin{pmatrix} 1-\alpha & 0 \\ \alpha & 1 \end{pmatrix} \qquad \mathcal{R} = \begin{pmatrix} 1 & \beta \\ 0 & 1-\beta \end{pmatrix}. \tag{8}$$

The phase diagram of this model shows two phases. For $\alpha < \beta$ the system is in a low-density phase with an average particle density $\rho = \alpha/2 < \frac{1}{2}$ whereas in the high-density phase $\alpha > \beta$ one has $\rho = 1 - \beta/2 > \frac{1}{2}$. The total current in the thermodynamic limit is given by $j = \min(\alpha, \beta)$. The physical behaviour is closely related to that of asymmetric exclusion models with continuous time evolution [4] (there is an additional phase with maximal density in the latter case). It plays a role in traffic models [9] as well as in polymer physics [11]. Related models with deterministic dynamics can be found in [12] and the influence of defects has been studied in [13].

As we will show below, the stationary state of the exclusion model (equations (7) and (8)) can be written as a matrix product state with alternating pairs of matrices (E, D) and $(\hat{E}, \hat{D})$ such that the probability of finding the system in the configuration $(\tau_1, \tau_2, \ldots, \tau_{2N})$

is given by

$$P_0(\tau_1, \tau_2, \ldots, \tau_{2N}) = Z^{-1} \langle W| \prod_{i=1}^{N} \Big[\big(\tau_{2i-1}\hat{D} + (1-\tau_{2i-1})\hat{E}\big) \big(\tau_{2i} D + (1-\tau_{2i})E\big) \Big] |V\rangle \,. \tag{9}$$

As in (2), we may rewrite this expression as a tensor product

$$\begin{aligned} |P_0\rangle &= Z^{-1} \langle W| \begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle \\ &= Z^{-1} \langle W| \left[\begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \right]^{\otimes N} |V\rangle \end{aligned} \tag{10}$$

where $Z = \langle W|\big((\hat{E}+\hat{D}) \otimes (E+D)\big)^{\otimes N}|V\rangle$. Obviously the mechanism which ensures the stationarity of $|P_0\rangle$ has to be different from the usual one for continuous time evolution. Instead of the additive cancellation from site to site we now need a multiplicative mechanism suitable for stationary states of parallel transfer matrices (which have the eigenvalue one rather than zero). In the case of the above exclusion model we found a very simple mechanism. The assumption is that in each time step the two pairs of matrices (E, D) and $(\hat{E}, \hat{D})$ are exchanged:

$$\begin{aligned} T_1|P_0\rangle &= Z^{-1} \langle W| \, T_1 \left[\begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \right]^{\otimes N} |V\rangle = Z^{-1} \langle W| \left[\begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} \right]^{\otimes N} |V\rangle \\ T_2 T_1|P_0\rangle &= Z^{-1} \langle W| \, T_2 \left[\begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} \right]^{\otimes N} |V\rangle = Z^{-1} \langle W| \left[\begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \right]^{\otimes N} |V\rangle \end{aligned} \tag{11}$$

so that $T|P_0\rangle = |P_0\rangle$. This exchange mechanism can be realized by the ansatz

$$\mathcal{T}\left[\begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} \right] = \begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \qquad \langle W|\mathcal{L}\begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} = \langle W|\begin{pmatrix} E \\ D \end{pmatrix} \qquad \mathcal{R}\begin{pmatrix} E \\ D \end{pmatrix}|V\rangle = \begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix}|V\rangle \tag{12}$$

which is equivalent to the algebra

$$\begin{aligned} &[E, \hat{E}] = [D, \hat{D}] = 0 \\ &E\hat{D} = [\hat{E}, D] \\ &\hat{D}E = 0 \end{aligned} \tag{13}$$

with the boundary conditions

$$\begin{aligned} &\langle W|\hat{E}(1-\alpha) = \langle W|E \qquad && (1-\beta)D|V\rangle = \hat{D}|V\rangle \\ &\langle W|(\alpha\hat{E} + \hat{D}) = \langle W|D \qquad && (E + \beta D)|V\rangle = \hat{E}|V\rangle \,. \end{aligned} \tag{14}$$

If this algebra has non-trivial representations, the above ansatz implies that the resulting matrix product state $|P_0\rangle$ is stationary. Again it must be emphasized that we have not solved the problem so far. By introducing the mechanism (11), (12) we only reformulated the original problem as a set of algebraic relations.

Before discussing representations let us first study the algebra (13), (14) on an abstract level. The commutation relations (13) involve four different objects $E, D, \hat{E}, \hat{D}$. However, only three of them are independent since the matrix product in (10) is invariant under the transformation

$$E \to U^{-1}E \qquad D \to U^{-1}D \qquad \hat{E} \to \hat{E}U \qquad \hat{D} \to \hat{D}U \,. \tag{15}$$

Because of $[E+D, \hat{E}+\hat{D}]=0$ it is possible to choose a basis in which both operators $E+D$ and $\hat{E}+\hat{D}$ are diagonal. Taking now $U=(E+D)^{1/2}(\hat{E}+\hat{D})^{-1/2}$ both operators become identical so that we may add the relation

$$C = E + D = \hat{E} + \hat{D}\,. \tag{16}$$

By eliminating $\hat{E}$ and E we therefore obtain a quadratic algebra of three independent objects C, D and $\hat{D}$. It is defined by three bulk equations

$$\hat{D}C = \hat{D}D = D\hat{D} \qquad [D-\hat{D}, C] = 0 \tag{17}$$

and two boundary relations

$$\langle W|\left(D - \alpha C - (1-\alpha)\hat{D}\right) = 0 \qquad \left((1-\beta)D - \hat{D}\right)|V\rangle = 0\,. \tag{18}$$

Matrix product states based on quadratic algebras with three objects were first studied in [7]. A detailed analysis of algebras with more than two objects and their representations will be given in [14].

It is important to prove that the simplified algebra (16), (17) already works on an abstract level. This means that as for the harmonic oscillator algebra, all physical quantities have to be given uniquely by the commutation relations without knowing anything about representations. In the case of a two-site chain this can be done easily by hand. It turns out that for $\alpha \neq \beta$ any expectation value of two operators is a given number times $Z = \langle W|CC|V\rangle$:

$$\langle W|\hat{D}C|V\rangle = \langle W|\hat{D}D|V\rangle = \frac{\alpha^2(1-\beta)}{(\alpha^2+\alpha\beta)(1-\beta)+\beta^2}\,Z \tag{19}$$

$$\langle W|CD|V\rangle = \frac{\alpha^2(1-\beta)+\alpha\beta}{(\alpha^2+\alpha\beta)(1-\beta)+\beta^2}\,Z\,. \tag{20}$$

Therefore the stationary state of a two-site chain reads

$$\begin{aligned} |P_0\rangle &= \frac{1}{Z}\,\{\langle W|\hat{E}E|V\rangle,\ \langle W|\hat{E}D|V\rangle,\ \langle W|\hat{D}E|V\rangle,\ \langle W|\hat{D}D|V\rangle\} \\ &= \frac{1}{\alpha^2+\alpha\beta-\alpha^2\beta+\beta^2-\alpha\beta^2}\,\{(1-\alpha)\beta^2,\ \alpha\beta,\ 0,\ (1-\beta)\alpha^2\}\,. \end{aligned} \tag{21}$$

In order to show that the same can be done for chains with more than two sites, we now prove that the expectation value of any sequence of operators is given uniquely by means of the commutation relations. For this purpose it is more convenient to use a different basis of operators which is defined by the invertible transformation

$$\begin{aligned} X &= \frac{1}{\alpha\beta}(D - \alpha C + (\alpha-1)\hat{D}) \\ Y &= \frac{1}{\alpha\beta}((1-\beta)D - \hat{D}) \\ S &= \frac{1}{\alpha\beta}(D - \hat{D})\,. \end{aligned} \tag{22}$$

In this basis, the bulk algebra (17) reads

$$[X,S] = [Y,S] = 0 \qquad YX = (1-\alpha)SY + (1-\beta)XS - (1-\alpha)(1-\beta)S^2 \tag{23}$$

and the boundary relations (18) become particularly simple:

$$\langle W|X = 0 \qquad Y|V\rangle = 0\,. \tag{24}$$

As can easily be seen, the application of the bulk relations (23) allows every product of $2N$ matrices X, Y and S to be ordered as a linear combination of terms like $X^n\,S^{2N-n-m}\,Y^m$. Since the only non-zero expectation values of these terms is $\langle W|S^{2N}|V\rangle$, the expectation value of any product of $2N$ matrices is a well defined number times $\langle W|S^{2N}|V\rangle$. The actual value of $\langle W|S^{2N}|V\rangle$ is irrelevant since it is cancelled by the normalization constant $Z = \langle W|C^{2N}|V\rangle$. Thus it is obvious that the algebra (23), (24) determines the ground state $|P_0\rangle$ *uniquely* on an abstract level. It should be emphasized that the mathematical structure of this algebra is different from those for exclusion models with continuous time evolution where one has linear terms in the bulk algebra (e.g. $DE = D + E$). Whereas in the latter case any expectation value can be reduced to the empty bracket $\langle W|V\rangle$, the algebra (17) does not allow a reduction in the number of factors in a matrix product. Rather, we have shown that by means of algebraic rules the expectation values of all words with the same number of factors are linearly dependent.

The algebra (17), (18) can be represented by two-dimensional matrices. For $\alpha \neq \beta$ a representation in which C is diagonal is given by

$$\begin{aligned} &C_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \qquad D_1 = \begin{pmatrix} \alpha & 0 \\ -\alpha\beta & \alpha\beta \end{pmatrix} \qquad \hat{D}_1 = \begin{pmatrix} \alpha(1-\beta) & 0 \\ -\alpha\beta & 0 \end{pmatrix} \\ &\langle W_1| = (\alpha, 1-\alpha) \qquad |V_1\rangle = \begin{pmatrix} 1-\beta \\ -\beta \end{pmatrix}. \end{aligned} \tag{25}$$

The normalization constant in this representation can be computed easily:

$$Z_1 = (1-\beta)\,\alpha^{2N+1} - (1-\alpha)\,\beta^{2N+1}\,. \tag{26}$$

By inserting these matrices into (10), the ground state can be computed immediately. Using this representation it is now easy to verify the result for a two-site chain which was given in equations (19)–(21).

As already mentioned, the matrix C acts like a transfer matrix between the points of density correlation functions. Therefore, the length scales to be expected are essentially given by the quotients of the eigenvalues of C. Thus in the present case the correlation functions involve only a single length scale, namely $\log^{-1}(\alpha/\beta)$. This length scale diverges at the phase transition line $\alpha = \beta$ where the constant Z_1 vanishes so that the above representation becomes singular. It turns out that in this case the operator C cannot be diagonalized so that one has to use a different representation where C has a Jordan normal form:

$$\begin{aligned} &C_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad D_2 = \begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix} \qquad \hat{D}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1-\alpha \end{pmatrix} \\ &\langle W_2| = (1, 0) \qquad |V_2\rangle = \begin{pmatrix} 1 \\ 1-\alpha \end{pmatrix}. \end{aligned} \tag{27}$$

Because of

$$C_2^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \tag{28}$$

the normalization constant Z is now linear in the system size:

$$Z_2 = 1 + 2N(1-\alpha)\,. \tag{29}$$

Using the matrix product formalism it is now easy to derive explicit expressions for equal-time correlation functions. Following the ideas of [10], we first compute the n-point functions of the operators

$$\eta_{2j} = \frac{\tau_{2j}-\alpha}{1-\alpha} \qquad \eta_{2j-1} = \frac{\tau_{2j-1}}{1-\beta}\,. \tag{30}$$

Denoting the corresponding matrices by

$$F_j = \begin{cases} (D-\alpha C)/(1-\alpha) & \text{if } j \text{ even} \\ \hat{D}/(1-\beta) & \text{if } j \text{ odd} \end{cases} \tag{31}$$

and assuming that the positions $j_1, \ldots, j_n$ are chosen in increasing order these correlation functions are given by

$$\langle \eta_{j_1}\eta_{j_2}\ldots\eta_{j_n}\rangle = \frac{1}{Z}\,\langle W|C^{j_1-1}\,F_{j_1}\,C^{j_2-j_1-1}\,F_{j_2}\,C^{j_3-j_2-1}\cdots C^{j_n-j_{n-1}-1}\,F_{j_n}\,C^{2N-j_n}|V\rangle\,. \tag{32}$$

Using the representations (25)–(29) it is easy to check that

$$F_j C^{k-j-1} F_k = F_j C^{k-j} \qquad (j<k) \tag{33}$$

so that the n-point correlation functions reduce to the one-point function $\langle\eta_j\rangle$:

$$\langle \eta_{j_1}\eta_{j_2}\ldots\eta_{j_n}\rangle = \langle\eta_{j_1}\rangle = Z^{-1}\,\langle W|C^{j_1-1}\,F_{j_1}\,C^{2N-j_1}|V\rangle\,. \tag{34}$$

For $\alpha\neq\beta$ the one-point function reads

$$\begin{aligned} \langle\eta_{2j}\rangle &= \frac{1}{Z_1}\,\alpha^{2N+1-2j}\,(1-\beta)\,(\alpha^{2j}-\beta^{2j}) \\ \langle\eta_{2j-1}\rangle &= \frac{1}{Z_1}\,\alpha^{2N+2-2j}\,\big(\alpha^{2j-1}(1-\beta)-\beta^{2j-1}(1-\alpha)\big) \end{aligned} \tag{35}$$

whereas at the transition line $\alpha=\beta$ we have

$$\begin{aligned} \langle\eta_{2j}\rangle &= \frac{1}{Z_2}\,2j\,(1-\alpha) \\ \langle\eta_{2j-1}\rangle &= \frac{1}{Z_2}\,\big(\alpha+(2j-1)(1-\alpha)\big)\,. \end{aligned} \tag{36}$$

Although we used the two-dimensional matrices at this point, equations (34)–(36) do not depend on the choice of the representation since we have shown that the expectation value of any sequence of operators is uniquely given by the commutation relations of the algebra.

Resubstituting τ_j into (34) we obtain an exact expression for the n-point density correlation functions $\langle\tau_{j_1}\tau_{j_2}\cdots\tau_{j_n}\rangle$. Assuming that $j_1<j_2<\cdots<j_n$ and denoting $\sigma_j = 1-(j \bmod 2)$ they are given by

$$\begin{aligned} \langle\tau_{j_1}\tau_{j_2}\ldots\tau_{j_n}\rangle = \alpha^n\prod_{i=1}^{n}\sigma_{j_i} + \sum_{k=1}^{n}\left(\prod_{i=1}^{k-1}\sigma_{j_i}\right)\alpha^{k-1}\,(1+(\beta-\alpha)\sigma_{j_k}-\beta) \\ \times\left(\prod_{i=k+1}^{n}(1-\beta+\beta\sigma_{j_i})\right)\langle\eta_{j_k}\rangle\,. \end{aligned} \tag{37}$$

As a special case this formula includes the one-point functions ($1\leqslant x\leqslant N$)

$$\begin{aligned} \langle\,\tau_{2x}\,\rangle &= \begin{cases} \alpha+(1-\alpha)\dfrac{1-(\frac{\beta}{\alpha})^{2x}}{1-\frac{1-\alpha}{1-\beta}(\frac{\beta}{\alpha})^{2N+1}} & \text{if } \alpha\neq\beta \\ \alpha+(1-\alpha)^2\dfrac{2x}{1+2N(1-\alpha)} & \text{if } \alpha=\beta \end{cases} \\ \langle\,\tau_{2x-1}\,\rangle &= \begin{cases} (1-\beta)\dfrac{1-\frac{1-\alpha}{1-\beta}(\frac{\beta}{\alpha})^{2x-1}}{1-\frac{1-\alpha}{1-\beta}(\frac{\beta}{\alpha})^{2N+1}} & \text{if } \alpha\neq\beta \\ (1-\alpha)^2\dfrac{2x-1}{1+2N(1-\alpha)}+\dfrac{\alpha(1-\alpha)}{1+2N(1-\alpha)} & \text{if } \alpha=\beta \end{cases} \end{aligned} \tag{38}$$

and the two-point correlation functions ($1 \leqslant x < y \leqslant N$)

$$
\begin{aligned}
&\langle \tau_{2x}\tau_{2y} \rangle = \langle \tau_{2x} \rangle - \alpha + \alpha\langle \tau_{2y} \rangle \\
&\langle \tau_{2x}\tau_{2y-1} \rangle = (1-\beta)(\langle \tau_{2x} \rangle - \alpha) + \alpha\langle \tau_{2y-1} \rangle \\
&\langle \tau_{2x+1}\tau_{2y} \rangle = \langle \tau_{2x+1} \rangle \\
&\langle \tau_{2x-1}\tau_{2y-1} \rangle = (1-\beta)\langle \tau_{2x-1} \rangle
\end{aligned}
\tag{39}
$$

which were given as conjectures in [10]. It is obvious that the whole problem has an alternating structure since the results for even and odd sites are different. This structure is due to the definition of the model and appears also in the ansatz (10) where different matrices are placed at even and odd sites.

The example of the asymmetric exclusion model shows that the powerful matrix product formalism can be applied successfully to models with parallel dynamics. Since models of this type are studied widely, it would be interesting to find further examples in order to understand under which conditions the matrix product technique can be applied. A first step is to consider a generalized version of the present model where particles diffuse stochastically with a given probability [15]. Since in the limit of a very low hopping probability sequential dynamics is recovered, this helps to understand the connection between the mechanisms for parallel and sequential models. Another interesting problem would be to solve the same model on a ring in the presence of a defect. From this one could learn how to solve the full exclusion process (with stochastic hopping in both directions) on a ring with a defect [16]. Despite intensive efforts, the exact solution to this problem is not yet known.

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